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Pricing Power Options Using the Actuarial Approach*

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Abstract: The problem of pricing options in financial engineering is discussed. Under the assumption that underlying assets have some jumps with stochastic amplitudes, the paper presents the insurance actuary price for a power option through the fair premium law. First, considering that the market information and its influence on prices of risk assets is stochastic, a jump-diffusion model with stochastic jump-size is introduced as a risky-asset price process. Second, a price formula for a power option is derived by using the properties of Poisson distributions and the total expectation formula. In the formula, fair price for options is given as a series. Therefore, investors can compute it easily and better manage the price risk.

Keywords: power options; jump-diffusion process; insurance actuary pricing; risk management

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1 Introduction

An option is an right, but not an obligation, to buy (sell) some underlying assets at a prescribed price (i.e. striking price). As an efficient risk-management instrument, options play an important role in current financial markets. Pricing options is one of the most important problems in financial engineering and mathematical finance. Models and methods for pricing are the keys to the problem. There are two classical pricing methods. The first is no-arbitrage replicate pricing. Black and Scholes[1], and Merton[2] relied on an ingenious no-arbitrage argument to price an option on a stock when the interest rate is constant and the stock price follows a geometric Brownian process. They presented a self-financing, dynamic trading strategy between the bond and stock accounts that replicates the payoff of the option. They then argued that the absence of arbitrage dictates that the option price is equal to the cost of setting up the replicating portfolio. The appeal of the argument lies in its reliance on the absence of arbitrage and existence of the replicating portfolio. Thus the argument should not be used in incomplete markets in which the no-arbitrage replicating portfolio does not always exist. The second approach is the martingale method. This approach consists of writing the value of the security as the expected value of the discounted payoff under an equivalent martingale measure and calculating this expectation using probabilistic methods. Although the martingale method is widely used to price derivatives today, it also has its drawbacks. When there is arbitrage (opportunity) in financial markets, the method is invalid because there is not any equivalent

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martingale measure. When the market is incomplete, there are infinite martingale prices, which all are no-arbitrage, because there are infinite equivalent martingale measures. In the latter, how to determine a trade price is also a complex problem. Bladt and Ryderg[3] introduced a insurance actuary pricing method through fair premium law. They proved in Black-Scholes model the celebrated Black-Scholes formula from view of insurance actuary. Subsequently the pricing method is widely applied in pricing options. Yan and Liu[4] considered insurance actuary and presented a explicit price formula in the case where the amplitudes of jumps are supposed to be deterministic time-function. But in real financial markets, amplitudes of jumps on stock price, which are affected by market information, always are stochastic. But in the model above the fact that jump amplitudes of risky assets are always not deterministic and should be stochastic for change in market information didn't be considered. Now finance trade is so active in financial markets that all kinds of options arise one after another. Options pay important roles in financial risk management. Power options are a class of European options. Because of its bigger elastic payoff than that of other options, power options get more and more attention from market investors and are widely applied for risk management. Thus pricing for power options plays an important role in financial risk analysis. In the present paper we suppose the stock price process admits a jump-diffusion model. This paper then studies insurance actuary pricing problem in the market where risky asset price admits a jump-diffusion process. Compared with the authors above, here the amplitudes of jumps are not deterministic but stochastic (admit a distribution). A price formula for a power option is deduced by using the properties of Poisson distributions and total expectation formula tactically. One highlight is that it can be computed and approximated easily and facilitate estimating option price, which is very important in financial practice.

2 Market model

Consider a continuous-time financial market in $[0,T]$. It can be described by a filtered complete probability space $\{\Omega, F, F_t, P\}$. $\{F_t\}_{0\le t\le T}$ is a natural σ-filtration generated by a Brownian motion W_t and a Poisson process N_t (also suppose that $\{F_t\}_{0\le t\le T}$ satisfies usual hypotheses). We denote by $\{F_t\}_{0\le t\le T}$ the valid market information that investors can know up to time t. We consider a market with two traded assets: a riskless money market account B^1 with instant yield r_t and a risky asset. We assume that the price S of the risky asset satisfies the following stochastic differential equation

$$\mathrm{d}S_t = S_{t-}(u_t\mathrm{d}t + \sigma_t\mathrm{d}W_t + \gamma_t J_t\mathrm{d}N_t), \tag{1}$$

where S_0 is deterministic, W_t is a standard Brownian motion; N_t is an (F_t, P)-Poisson jump process, independent of W_t, with deterministic and bounded intensity $\lambda_t \ge 0$; u_t, σ_t, γ_t are deterministic and bounded functions of t, J_t are independent, identically distributed random variables (i.i.d. in short).

In this paper, we assume that $\gamma_t = \gamma$ is a constant, J_t admits two-point distribution

$$\begin{cases} P\{J_t = u\} = p, \\ P\{J_t = d\} = 1 - p. \end{cases} \tag{2}$$

We also assume γu, $\gamma d > -1$ (by this, we ensure that the asset price is positive).

Remark 1 In real financial markets, there always exist some large jumps in risky assets when significant new information or catastrophic events arise. Between two jumps risky asset price fluctuates smoothly. The kind of price change can be characterized by jump-diffusion models perfectly. The jump induced by significant new information or catastrophic events is characterized by N_t. The smooth fluctuation of risky asset price can be modeled by a Browian motion W_t. Since jump-size is always related to all kinds of market information, it should be stochastic. Many existed papers[4,5] are concerned the model with deterministic jump-size which obviously cannot capture actual jumps of asset price. The assumption that jump-size admits two-point distribution improves the shortcoming in some sense.

3 Pricing power options by insurance actuary method

3.1 Insurance actuary method for pricing

It is well known that martingale method is not suitable for arbitrage markets and that there are infinite martingale prices in an incomplete market. To overcome these disadvantages, Bladt and Rydberg[3] introduced the notion of insurance actuary price. The problem of pricing options is changed to the problem of insure through fair premium law. The basic conception is "buy a option, counterpart (i.e. option writer) will undertake some risk before expiration time of the option. If he wishes to insure against the risk, the premium is the fair price of the option, that is the value of options is measured in term of the risk that the counterpart will undertake". In the paper we adopt the notion of insurance actuary price in Bladt and Rydberg[3], Xiong[5], the key point is that risky asset price is discounted by expected yield and that risk-free asset is discounted by risk-free compounded rate of interest and that we calculate expectation under real-life probability.

Definition 1 Let the price process of risky asset is S_t, $\frac{E(S_t)}{S_0}$ is said to be expected yield of S_t on $[0,t]$. Obviously, if β_s is the instant compounded expected yield of S_t, then

$$\exp\Big(\int_0^t \beta_s \mathrm{d}s\Big) = \frac{E(S_t)}{S_0}.$$

Definition 2 Let that underlying asset is S and that $H(S_T, K)$ is a European option with striking price K and terminative time T. Define the insurance actuary price V of $H(S_T, K)$ as

$$V = E\Big[H\Big(\exp\Big(-\int_0^T \beta_s \mathrm{d}s\Big)S_T, \exp\Big(-\int_0^T r_s \mathrm{d}s\Big)K\Big)\Big], \tag{3}$$

where β_s is the instant compounded expected yield of S_t, r_s is the risk-free instant compounded rate of interest.

Remark 2 Compared with martingale method, insurance actuary pricing method works under the conditions that risky asset (risk-free asset) price is discounted by expected payoff yield (risk-free compounded rate of interest) and that we calculate expectation under the real-life probability.

3.2 Insurance actuary price of a power option

A power option is an European option and its terminative payoff is $(S_T^a - K)^+$. According insurance actuary pricing method, insurance actuary price of the power option can be denoted as

$$C_m \hat{=} E\Big[\Big(\Big(\exp\Big(-\int_0^T \beta_s \mathrm{d}s\Big)S_T\Big)^a - \exp\Big(-\int_0^T r_s \mathrm{d}s\Big)K\Big)^+\Big]. \tag{4}$$

For computing insurance actuary price of a power option, we start with calculating the expected yield

$$\frac{E(S_T)}{S_0} = \exp\Big(-\int_0^T \beta_s \mathrm{d}s\Big).$$

Theorem 1 With the model assumptions (1) and (2), the expected yield on $[0,T]$ of the risky asset S_t is

$$\frac{E(S_T)}{S_0} = \exp\Big(\gamma\, up \int_0^T (\lambda_s + u_s)\mathrm{d}s\Big). \tag{5}$$

Proof From the Doleans-Dade stochastic exponential formula[6] we have

$$S_T = S_0 \prod_{n=1}^{N_T}(1+\gamma J_n)\exp\Big[\int_0^T \Big(u_s - \frac{1}{2}\sigma_s^2\Big)\mathrm{d}s\Big] \times \exp\Big(\int_0^T \sigma_s \mathrm{d}W_s\Big). \tag{6}$$

Consequently, the total expectation formula yields the following result

$$\begin{aligned} E\Big(\prod_1^{N_T}(1+\gamma J_n)\Big) &= E\Big[E\Big(\prod_1^{N_T}(1+\gamma J_n)\,|\,N_T\Big)\Big] = \sum_{n=1}^{\infty} E\Big(\prod_1^{n}(1+\gamma J_n)\Big)P(N_T=n) \\ &= \exp\Big(-\int_0^T \lambda_s \mathrm{d}s\Big)\exp\Big\{\int_0^T \lambda_s \mathrm{d}s[1+\gamma E(J_n)]\Big\} \\ &= \exp\Big[\gamma E(J_n)\int_0^T \lambda_s \mathrm{d}s\Big] = \exp\Big(\gamma\, up \int_0^T \lambda_s \mathrm{d}s\Big), \end{aligned}$$

the fourth equality holds because J_n is i.i.d..

Thus

$$\begin{aligned} \frac{E(S_T)}{S_0} &= E\Big\{\prod_{n=1}^{N_T}(1+\gamma J_t)\exp\Big[\int_0^T \Big(u_s - \frac{1}{2}\sigma_s^2\Big)\mathrm{d}s\Big] \times \exp\Big(\int_0^T \sigma_s \mathrm{d}W_s\Big)\Big\} \\ &= \exp\Big(\gamma\, up \int_0^T \lambda_s \mathrm{d}s + \int_0^T u_s \mathrm{d}s\Big) = \exp\Big(\gamma\, up \int_0^T (\lambda_s + u_s)\mathrm{d}s\Big) \hat{=} \beta, \end{aligned}$$

the last equality holds because

$$\exp\Big[\int_0^t \sigma_s \mathrm{d}W_s - \int_0^t \frac{1}{2}\sigma_s^2 \mathrm{d}s\Big]$$

is a martingale (see [5] for detail).

Since the jump-size of stock price is stochastic, we need the following lemma to compute option price.

Lemma 1 Let c, b are constant

$$P(V_n = c) = 1, \quad P(V_n = b) = 0, \quad G_T = \sum_{n=1}^{N_T} V_n.$$

Let that N^1 is independent of N^2 and that N^1, N^2 admit Poisson distributions with parameters $p\int_0^T \lambda_s \mathrm{d}s$ and $(1-p)\int_0^T \lambda_s \mathrm{d}s$, respectively. Then law (M_t) = law (cN^1+bN^2).

Proof Let L_n satisfies

$$\begin{cases} P(L_n=1)=p, \\ P(L_n=0)=1-p. \end{cases}$$

Then

$$G_T=\sum_{n=1}^{N_T} V_n = cN_T+(b-c)\sum_{n=1}^{N_T} L_n = cN^1+bN^2. \tag{7}$$

Recall that $\sum\limits_{n=1}^{N_T} L_n$ admits a Poisson distributions with parameters $(1-p)\int_0^T \lambda_s \mathrm{d}s$, so that Lemma 4 holds.

Corollary 1 $\sum\limits_{n=1}^{N_T} \ln(1+\gamma J_n)$ and $\ln(1+\gamma u)N^1+\ln(1+\gamma d)N^2$ admit the same distributions.

Theorem 2 Let $\Phi(\cdot)$ denotes the standard normal distribution function. With the model assumptions (1) and (2), the insurance actuary price of a power option can be denoted as

$$C_m=\sum_{k_1,k_2} C^{k_1,k_2}\exp\Big(-\int_0^T \lambda_s \mathrm{d}s\Big)\frac{(\int_0^T \lambda_s \mathrm{d}s)^{k_1+k_2}p^{k_1}(1-p)^{k_2}}{k_1!\, k_2!}, \quad k_1,k_2\in N, \tag{8}$$

where

$$C^{k_1,k_2}=\beta^a\exp\Big(\frac{a^2(\sigma_T)^2+2a}{2}\Big)\Phi\Big(\frac{(m_T+a(\sigma_T)^2-l)}{\sigma_T}\Big)-K\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)\Phi\Big(\frac{m_T-l}{\sigma_T}\Big),$$

$$M_T=\ln S_0+\ln(1+\gamma u)k_1+\ln(1+\gamma d)k_2+\int_0^T\Big(u_s-\frac{1}{2}\sigma_s^2\Big)\mathrm{d}s,$$

$$\sigma_T=\sqrt{\int_0^T(\sigma_s)^2\mathrm{d}t}; \qquad l=\frac{\ln K-\int_0^T r_s\mathrm{d}s-\exp(\gamma\, up\int_0^T \lambda_s+u_s\mathrm{d}s)}{a}.$$

Proof By equality (6), we have

$$\ln S_T=\ln S_0+\sum_{n=1}^{N_T}\ln(1+\gamma J_n)+\int_0^T\Big(u_s-\frac{1}{2}\sigma_s^2\Big)\mathrm{d}s+\int_0^T\sigma_s\mathrm{d}W_s^Q.$$

Recall Corollary 1: $\sum\limits_{n=1}^{N_T}\ln(1+\gamma J_n)$ and $\ln(1+\gamma u)N^1+\ln(1+\gamma d)N^2$ admit the same distributions, so

$$\ln S_T=\ln S_0+\ln(1+\gamma u)N^1+\ln(1+\gamma d)N^2+\int_0^T\Big(u_s-\frac{1}{2}\sigma_s^2\Big)\mathrm{d}s+\int_0^T\sigma_s\mathrm{d}W_s^Q.$$

Let

$$M_T=\ln S_0+\ln(1+\gamma u)k_1+\ln(1+\gamma d)k_2+\int_0^T\Big(u_s-\frac{1}{2}\sigma_s^2\Big)\mathrm{d}s, \quad \sigma_T=\sqrt{\int_0^T(\sigma_s)^2\mathrm{d}t}.$$

Consequently, when $N^1=k_1,\ N^2=k_2$, we have

$$\ln S_T\sim N\big(M_T,(\sigma_T)^2\big), \tag{9}$$

where $N(a,b)$ denotes a normal distribution with mean a and variance b.

Write C_m as C^{k_1,k_2} when $N^1=k_1,\ N^2=k_2$, then

$$\begin{aligned} C^{k_1,k_2} &= E\Big[\Big(\Big(\exp\Big(-\int_0^T\beta_s\mathrm{d}s\Big)S_T\Big)^a-\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)K\Big)^+\,\Big|\,N^1=k_1,\ N^2=k_2\Big] \\ &= E\Big[(\beta S_T)^a1_A-\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)K1_A\,\Big|\,N^1=k_1,\ N^2=k_2\Big], \end{aligned}$$

where

$$\begin{aligned} A &= \Big\{(\beta S_T)^a>\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)K\Big\} \\ &= \Big\{\ln S_T>\frac{\ln K-\int_0^T r_s\mathrm{d}s-\beta}{a}\Big\}\,\widehat{=}\,\{\ln S_T>l\}. \end{aligned}$$

By (9), we have

$$\begin{aligned} &E\big[(\beta S_T)^a1_A\,\big|\,N^1=k_1,\ N^2=k_2\big] \\ &=\beta^aE\big[\exp(a\ln S_T)1_A\,\big|\,N^1=k_1,\ N^2=k_2\big] \\ &=\beta^a\int_A\Big[\exp(ax)\frac{1}{\sqrt{2\pi}\sigma_T}\exp\Big(-\frac{(x-m_T)^2}{2(\sigma_T)^2}\Big)\Big]\mathrm{d}x \\ &=\exp\Big(\frac{a^2(\sigma_T)^2+2a}{2}\Big)\beta^a\int\limits_{x>l}\frac{1}{\sqrt{2\pi}\sigma_T}\exp\Big[\Big(-\frac{(x-(m_T+a(\sigma_T)^2))^2}{2(\sigma_T)^2}\Big)\Big]\mathrm{d}x \\ &=\beta^a\exp\Big(\frac{a^2(\sigma_T)^2+2a}{2}\Big)\Phi\Big(\frac{(m_T+a(\sigma_T)^2-l)}{\sigma_T}\Big), \end{aligned}$$

$$\begin{aligned} &E\Big[\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)K1_A\,\Big|\,N^1=k_1,\ N^2=k_2\Big] \\ &=\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)KE\big[1_A\,\big|\,N^1=k_1,\ N^2=k_2\big] \\ &=\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)KE\big[1_{(\ln S_T>l)}\,\big|\,N^1=k_1,\ N^2=k_2\big] \\ &=K\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)\int\limits_{x>\frac{l-m_T}{\sigma_T}}\frac{1}{\sqrt{2\pi}\sigma_T}\Big[\exp\Big(-\frac{x^2}{2}\Big)\Big]\mathrm{d}x \\ &=K\exp\Big(-\int_0^T r_s\mathrm{d}s\Big)\Phi\Big(\frac{m_T-l}{\sigma_T}\Big). \end{aligned}$$

Recall the law of iterated expectation, then

$$\begin{aligned} C_m &= E\Big[\Big(\Big(\exp\Big(-\int_0^T \beta_s \mathrm{d}s\Big)S_T\Big)^a - \exp\Big(-\int_0^T r_s \mathrm{d}s\Big)K\Big)^+\Big] \\ &= \sum_{k_1,k_2} C^{k_1,k_2} P(N^1 = k_1, N^2 = k_2) \\ &= \sum_{k_1,k_2} C^{k_1,k_2} P(N^1 = k_1) P(N^2 = k_2) \\ &= \sum_{k_1,k_2} C^{k_1,k_2} \exp\Big(-\int_0^T \lambda_s \mathrm{d}s\Big) \frac{(\int_0^T \lambda_s \mathrm{d}s)^{k_1+k_2} p^{k_1}(1-p)^{k_2}}{k_1!\, k_2!}. \end{aligned}$$

The proof is completed.

Remark 3 In the case $a = 1$, the insurance actuary price of an European call options is given by Theorem 6.

Remark 4 In the case $\gamma = 0$, we get the insurance actuary price of a power option in the B-S model. It coincide with martingale price and is arbitrage-free price.

Remark 5 Compared with usual price formula, option price is given as an infinity sum and not as an integrable with respect to an abstract measure in the price formula. Thus we can compute price for options easily, which is very important in practice and is the highlight of the price formula.

4 Conclusions

Pricing options is an important issue in mathematical finance. Recently, some scholars made some useful work such as [7-9]. This paper discusses the problem of pricing power options by insurance actuary method. We improve the drawback (jump-size is deterministic) in the existed paper on asset pricing through presenting a jump-diffusion model with stochastic jump-size. A pricing formula is given by using the properties of Poisson and total expectation formula. In practice investors can estimate coefficients σ_t, u_t, γ_t from historical data. Then the value of options can be determined by inserting them into pricing formula. The price is important reference to investors in buying and/or selling options. It is important to point out the pricing method and strategy of change in this paper can be extend to pricing other European options and exotic options.

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幂型期权的保险精算定价

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摘 要: 讨论了金融工程中的期权定价问题。在基础资产价格具有随机幅度跳跃的假设下，利用公平保费原理给出了幂型期权的保险精算价格。考虑到市场信息对风险资产价格影响的随机性，首先用一个具有随机跳跃幅度的跳扩散模型来刻画风险资产的价格过程，然后利用Poisson分布的性质和全期望公式导出了幂型期权的定价公式。在这个价格公式中公平的期权价格通过级数来表示，投资者因而可以容易地计算出期权价格，更好地管理价格风险。

关键词: 幂型期权；跳扩散过程；保险精算定价；风险管理